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COMPLETE SYSTEMS OF CONSERVATION LAWS FOR TWO-LAYER SHALLOW WATER MODELS

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A well-posedness criterion for a complete system of conservation laws is proposed that assumes maximum compatibility of the convexity domain of the closing conservation law with the domain of hyperbolicity of the model used. This criterion is used to obtain well-posed complete systems of conservation laws for the models of two-layer shallow water with a free-surface (model I*) and with a rigid lid (model* II*).*

Ovsyannikov [1] derived and studied three differential models of two-layer shallow water: model I, in which the upper boundary of the fluid is a free-surface, model II, in which it is a rigid lid, and model III, which is the general limiting case of the first two models. For these models, the hyperbolicity domains in which discontinuous solutions are possible were determined. This led to the problem of formulation of these models as complete systems of conservation laws [2–4].

In the present paper, we propose a solution of this problem using a well-posedness criterion for a complete system of conservation laws. This criterion assumes maximum compatibility of the convexity domain of the closing conservation law (which is the law of conservation of total energy) with the hyperbolicity domain of the model considered. For model I, the well-posed complete system contains, as the basis laws, the laws of conservation of mass in the layers, total momentum, and velocity jump at the interface between the layers. For model II, the basis of well-posed system (corresponding to flows for which the ratio of the depths of the layers is not too small) is a complete system in which mass, total momentum, and local momentum jump at the interface between the layers are conserved at discontinuities. At the same time, when the ratio of the depth of the upper layer to that of the lower layer is sufficiently small, the more well-posed system for model II is a complete system in which the law of conservation of local momentum in the lower layer is satisfied at discontinuities (along with the laws of conservation of mass and total momentum). Conversely, when the ratio of the depth of the lower layer to that of the upper layer is sufficiently small, the more well-posed system for model II is a complete system for which the law of conservation of local momentum in the upper layer is satisfied at discontinuities (along with laws of conservation of mass and total momentum).

1. Complete System of Conservation Laws in the General Case. We consider the following quasilinear system of conservation laws [2–4]:

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0, \tag{1.1}$$

where $\boldsymbol{u}(t,x) = (u_1,\ldots,u_m)$ is the desired piecewise-continuous vector function and $\boldsymbol{f}(\boldsymbol{u}) = (f_1,\ldots,f_m)$ is a specified smooth vector function. System (1.1) is called a complete system if there exists a scalar function $U(\boldsymbol{u})$ such that:

(a) its gradient $\boldsymbol{v}(\boldsymbol{u}) = U_{\boldsymbol{u}}$ is an integrating factor for system (1.1), i.e., $U_{\boldsymbol{u}} \cdot \boldsymbol{f}_{\boldsymbol{u}} = F_{\boldsymbol{u}}$ and, as a result, system (1.1) admits the additional *closing* conservation law

$$U(\boldsymbol{u})_t + F(\boldsymbol{u})_x = 0; \tag{1.2}$$

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(b) the map $\boldsymbol{u} \to \boldsymbol{v}(\boldsymbol{u})$ is locally one-to-one.

Introducing the *generating functions* (or *potentials*) $\Phi = \boldsymbol{v} \cdot \boldsymbol{u} - U$ and $\Psi = \boldsymbol{v} \cdot \boldsymbol{f} - F$ by a Legendre transformation, we write the complete system of conservation laws (1.1) in *symmetric* form

$$(\Phi_{\boldsymbol{v}})_t + (\Psi_{\boldsymbol{v}})_x = 0, \tag{1.3}$$

whose extended *nondivergent* form is

$$A\boldsymbol{v}_t + B\boldsymbol{v}_x = 0, \tag{1.4}$$

where $A = \Phi_{\boldsymbol{v}\boldsymbol{v}}$ and $B = \Psi_{\boldsymbol{v}\boldsymbol{v}}$ are symmetric matrices. In [5], it was pointed out that a wide class of equations of mathematical physics (in particular, gas-dynamic equations) can be brought to the symmetric form (1.3), (1.4). The fact that the completeness of the system (1.1) is equivalent to the possibility of writing it in the symmetric form (1.3) was shown in [5–7].

If the complete system (1.1) admits the closing conservation law (1.2) with the convex function $U(\boldsymbol{u})$, then the map $\boldsymbol{u} \to \boldsymbol{v}(\boldsymbol{u})$ of the *basis variables* $\boldsymbol{u}$ to the *canonical* variables $\boldsymbol{v}$ is automatically one-to-one, the potential $\Phi(\boldsymbol{v})$ is convex, and the matrix $A(\boldsymbol{v})$ entering in the symmetric system (1.4) is positive definite [the latter implies that system (1.1) is hyperbolic]. In [7, 8], the closing conservation law (1.2) for this complete system (1.1) (called a *convex extension* [7]) was used for selection of stable discontinuous solutions. These solutions are defined as solutions that in a weak sense [2] satisfy the *entropy inequality*

$$U(\boldsymbol{u})_t + F(\boldsymbol{u})_x \leqslant 0 \tag{1.5}$$

(the functions U and F are called an *entropy function* and *entropy flux*). Friedrichs and Lax [7] showed that, for the complete system (1.1) with the convex extension (1.2), inequality (1.5) is satisfied by solutions of the system with linear viscosity $\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = \mu \boldsymbol{u}_{xx}$ in the limit $\mu \to 0$. Lax [2] proved that if this complete system is strongly nonlinear, the entropy condition (1.5) is locally (i.e., for shock waves of rather small intensity) equivalent to the characteristic stability condition introduced in [9].

In many papers (see, e.g., [8, 10, 11]) it is assumed that for complete systems of conservation laws (1.1) with the convex extension (1.2) the entropy condition (1.5) ensures the unique global solvability of the Cauchy problem in a certain class of piecewise-continuous functions. Thus, the existence of the convex extension (1.2) is regarded as the key requirement for well-posed formulation of a hyperbolic system in the form of a complete system of conservation laws. For a particular complete system in the form (1.1), this means that the convexity domain Ω^c of its entropy function must be maximum compatible with its hyperbolicity domain Ω^h. In this case, $\Omega^c \subseteq \Omega^h$ since the complete system with convex extension is hyperbolic. In particular, if we have two different complete systems of conservation laws (1.1) with convexity domains Ω^c_1 and Ω^c_2 that are obtained from the same differential hyperbolic system and if $\Omega^c_2 \subset \Omega^c_1$, then the system with the convexity domain Ω^c_1 should be considered more well-posed. Below, complete systems of conservation laws for the equations of single-layer shallow water and two-layer shallow water are analyzed from this viewpoint.

2. Complete Systems of Conservation Laws for the Single-Layer Shallow Water Equations. Ignoring friction and assuming that the flow domain is a channel with a rectangular cross section and horizontal bottom, we write differential equations of single-layer shallow water theory in the form [4, 12]

$$h_t + q_x = 0, \tag{2.1}$$

$$v_t + (v^2/2 + gh)_x = 0, \tag{2.2}$$

where h is the fluid depth, $q = hv$ is the discharge rate, v is velocity, and g is the acceleration of gravity. Equations (2.1) and (2.2) are the laws of conservation of mass and local momentum for each fluid particle along the streamline. System (2.1), (2.2), as any other system of two scalar conservation laws, has an infinite number of other linearly independent conservation laws but only the following two of them have a physical meaning: the law of conservation of total momentum

$$q_t + (qv + gh^2/2)_x = 0 \tag{2.3}$$

and the law of conservation of total energy

$$e_t + [(v^2 + 2gh)q]_x = 0, \tag{2.4}$$

where

$$e = qv + gh^2. \tag{2.5}$$

Usually (see [4, 12]), when the differential model of shallow water (2.1)–(2.4) is formulated in the form of a complete system of conservation laws, the law of conservation of mass (2.1) and the law of conservation of total momentum (2.3) are used as basis conservation laws, and the law of conservation of total energy (2.4) is employed as the closing conservation law (the corresponding symmetric form of system (1.3) is given in [4]). In this complete system (denoted by S_1), the total energy (2.5), which in terms of the basis variables $\boldsymbol{u} = (h, q)$ is written as the function $e(h,q) = q^2/h + gh^2$, which is convex for $h > 0$, plays the role of the entropy function $U(\boldsymbol{u})$. The complete system S_1 thus has the convex extension (2.4) in the entire hyperbolicity domain of system (2.1), (2.3).

In contrast to the system S_1, the complete system consisting of the basis conservation laws (2.1), (2.2) and the closing conservation law (2.4) (this system is denoted by S_2) has the entropy function

$$e(h, v) = (v^2 + gh)h, \tag{2.6}$$

which is convex only for subcritical flows $|v| < \sqrt{gh}$. Therefore (according to the criterion proposed above), the system S_1 is more well-posed than S_2 (it is interesting that, in spite of this, for both systems S_1 and S_2 the entropy criterion (1.5) and the characteristic criterion [9] of stability are equivalent in the entire hyperbolicity domain $h > 0$). Finally, if Eqs. (2.1), (2.2) are taken as the basis conservation laws and the law of conservation of total momentum (2.3) is taken as the closing conservation law, then the entropy function $q(h, v) = h \cdot v$ for such a "totally unphysical" system is not convex for all values of the basis variables h and v. Thus, for the well-studied model of single-layer shallow water the criterion proposed in Sec. 1 uniquely determines the physically well-posed complete system (2.1), (2.3), (2.4).

3. Complete Systems of Conservation Laws for Two-Layer Shallow Water Equations. The differential equations of two-layer shallow water are [1]

$$h_t + q_x = 0, \qquad H_t + Q_x = 0; \tag{3.1}$$

$$v_t + [v^2/2 + g(h + H)]_x = -p_x, \qquad V_t + [V^2/2 + g(H + \lambda h)]_x = -\lambda p_x, \tag{3.2}$$

where h, $q = hv$, and v are the depth, discharge rate, and velocity in the upper layer, H, $Q = HV$, and V are the same variables in the lower layer, g is the acceleration of gravity, p is the pressure on the upper boundary, and $\lambda < 1$ is the ratio of the densities of the upper and lower layers. Equations (3.1) and (3.2) are the conservation laws for mass and local momentum in each layer.

From system (3.1), (3.2), for $p_x = 0$, we obtain model I with a free surface, and for

$$h + H = H^*, \qquad q + Q = 0, \qquad H^* = \text{const}, \tag{3.3}$$

we have model II with a rigid lid. System (3.1), (3.2) has two other physically meaningful conservation laws: for the total momentum,

$$\alpha_t + [QV + \lambda qv + g\varphi/2]_x = -\lambda H^* p_x, \tag{3.4}$$

where $\alpha = Q + \lambda q$, $\varphi = H^2 + \lambda h^2 + 2\lambda hH$, and for the total energy,

$$e_t + [QV^2 + \lambda qv^2 + 2g(H\alpha + \lambda h(Q + q))]_x = 0, \tag{3.5}$$

where

$$e = QV + \lambda qv + g\varphi. \tag{3.6}$$

There are no other independent conservation laws for model I.

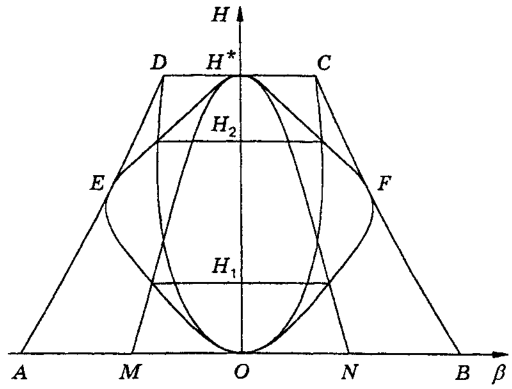

Fig. 1

Similarly to the single layer case (see [13]), we assume that for discontinuous waves, the total momentum α is conserved and the total energy e dissipates. Taking account of this, we choose the law of conservation of total energy (3.5) as the closing conservation law for both models I and II.

Model II. It follows from (3.3) that system (3.1), (3.2) must have only two basis conservation laws, one of which is the law of conservation of mass (3.1). To obtain the second law, it is necessary to eliminate the pressure p_x from the momentum equations (3.2) and (3.4), and this can be done by three different methods.

System II.1. Eliminating p_x from the local momentum equations (3.2), we obtain *the law of conservation of jump in local momentum at the interface between the layers* as the second basis conservation law:

$$\beta_t + [(V^2 - \lambda v^2)/2 + \mu g H]_x = 0, \tag{3.7}$$

where $\beta = V - \lambda v$ and $\mu = 1 - \lambda$ (we note that β coincides with the "normalized" velocity introduced in [1]). For the complete system thus obtained (which is denoted by II.1), the conservation laws (3.2) are not satisfied at a discontinuity because the jump in pressure p_x at the discontinuity must be determined from the law of conservation of total momentum (3.4).

Expressing the total energy (3.6), which, in view of (3.3), has the form

$$e = \frac{a}{Hh} Q^2 + g\varphi, \tag{3.8}$$

in terms of the basis variables β and H, we obtain

$$e(\beta, H) = \frac{Hh}{a} \beta^2 + g\varphi, \tag{3.9}$$

where $a = h + \lambda H$, $\varphi = H^2 + \lambda h^2 + 2\lambda h H$, and $h = H^* - H$. Function (3.9) is convex provided that

$$\left[1 + \frac{(h^2 - \lambda H^2)^2}{\lambda H h (H^*)^2}\right] \beta^2 < \frac{\mu g a^3}{\lambda (H^*)^2}. \tag{3.10}$$

This strengthens the hyperbolicity condition for model II obtained in [1]:

$$\beta^2 < \frac{\mu g a^3}{\lambda (H^*)^2}. \tag{3.11}$$

In Fig. 1, the hyperbolicity domain (3.11) in the plane of the basis variables β and H is shown by the curvilinear trapezium $ABCD$, whose vertices have the coordinates $A(-\beta_1, 0)$, $B(\beta_1, 0)$, $C(\beta_2, H^*)$, and $D(-\beta_2, H^*)$, where $\beta_1 = \sqrt{\mu g H^*/\lambda}$, $\beta_2 = \lambda\sqrt{\mu g H^*}$, and $\lambda = 0.5$. The convexity set of the total energy (3.10) is represented by the set OFH^*E inscribed in the trapezium, and its boundary points F and E have the coordinates $F(\beta_3, H_0)$ and $E(-\beta_3, H_0)$, where $H_0 = H^*/(1 + \sqrt{\lambda})$ and $\beta_3 = \sqrt{\mu g \sqrt{\lambda} H^*}$. One can see in Fig. 1 that the convexity domain (3.10) is in fairly good agreement with the hyperbolicity domain (3.11): it contains the main central part of the hyperbolicity domain and "cuts off" its corners, which correspond to physically less stable flows with a rather large jump of momentum at the interface between the layers in the case where the depth of one of them is small.

System II.2. Assuming that the local momentum of the lower layer is conserved at a discontinuity and eliminating p_x from Eq. (3.4) and from the second of Eqs. (3.2), we obtain the basis conservation law

$$\bar\beta_t + \left[\frac{v^2}{2h}(hH^* - 2aH) + \frac{g}{2}(H^2 + 2Hh + \lambda h^2)\right]_x = 0,$$

where $\bar\beta = H^*V - \alpha = h\beta$. This basis conservation law follows from the laws of conservation of total momentum (3.4) and local momentum in the lower layer [the second of Eqs. (3.2)] provided that the pressure jump p_x at a discontinuity is determined from one of them.

Writing the total energy (3.8) in the new basis variables $\bar\beta$ and H, we obtain the function $e(\bar\beta, H) = H\bar\beta^2/(ha) + g\varphi$, which is a convex function provided that

$$\beta^2 < \mu g a^3/\psi, \tag{3.12}$$

where $\psi = [r^3 + (2+\lambda)r^2 + (1+2\lambda)r + \lambda]H^2$ $(r = h/H)$. From Fig. 1, where the convexity domain (3.12) is bounded by the curve passing through the points D, O, and C, it follows that this domain is in good agreement with the hyperbolicity domain (3.11) only for rather large depths of the lower layer H (more accurately, for small $r = h/H$). Then, according to the well-posedness criterion of complete systems of conservation laws proposed above, system II.1 is more well-posed than system II.2 for depths $0 < H < H_2$ with $H_2 = (3 - \sqrt{1+8\lambda})H^*/(2\mu)$, for which the convexity domain (3.12) corresponding to system II.2 is strictly inside the convexity domain (3.10) corresponding to system II.1 (see Fig. 1). Conversely, for depths $H_2 < H < H^*$, for which the convexity domain (3.10) is strictly inside the domain (3.12), system II.2 is more well-posed than system II.1.

System II.3. Assuming that the local momentum of the upper layer is conserved at a discontinuity and eliminating p_x from Eq. (3.5) and from the first of Eqs. (3.2), we obtain

$$\tilde\beta_t + \left[\frac{v^2}{2H}(2ah - \lambda HH^*) + \frac{g}{2}(H^2 - 2\lambda HH^* - \lambda h^2)\right]_x = 0, \tag{3.13}$$

where $\tilde\beta = \alpha - \lambda H^* v = H\beta$. The new basis conservation law (3.13) is an integral consequence of the laws of conservation of total momentum and local momentum in the upper layer provided that the pressure jump p_x at a discontinuity is determined from one of them.

Expressing the total energy (3.8) in terms of the basis variables $\tilde\beta$ and H, we obtain the function

$$e(\tilde\beta, H) = \frac{h}{Ha}\tilde\beta^2 + g\varphi,$$

which is convex provided that

$$\beta^2 < \mu g a^3/\psi_1, \tag{3.14}$$

where $\psi_1 = [\lambda^2 R^3 + \lambda(3+\mu)R^2 + (3+2\mu)R + (1+2\mu)]h^2$ $(R = H/h)$. From Fig. 1, where the convexity domain (3.14) is bounded by the curve passing through the points M, H^*, and N [the abscissas of the points N and M are $\pm\sqrt{\mu g H^*/(1+2\mu)}$] it follows that the domain (3.14) is in relatively good agreement with the hyperbolicity domain only for sufficiently small depths of the lower layer H (more precisely, for small $R = H/h$). A comparison of the convexity domains for systems II.1 and II.3 shows that system II.1 is more well-posed than system II.3 for $H_1 < H < H^*$, where $H_1 = R_1 H^*/(1 - R_1)$ and R_1 is the corresponding root of the cubic equation $\lambda(2+\mu)R^3 + (3+2\mu)R^2 + 3\mu R - 1 = 0$, and, conversely, system II.3 is more well-posed than system II.1 for small depths of the lower layer $0 < H < H_1$.

Summarizing the results obtained above, we conclude that inside the hyperbolicity domain of system II there are three different subdomains and each of them has its own well-posed complete system of conservation laws. System II.1 is well-posed in the main central part of the hyperbolicity domain, where $H_1 < H < H_2$.

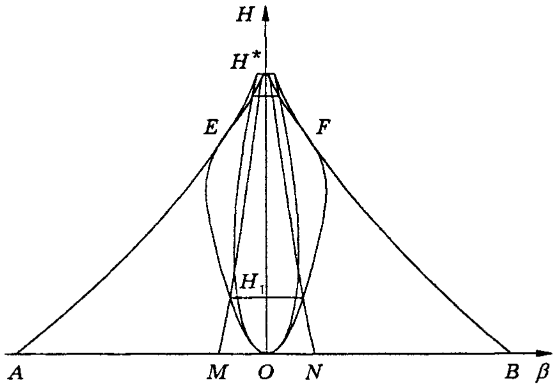

Fig. 2

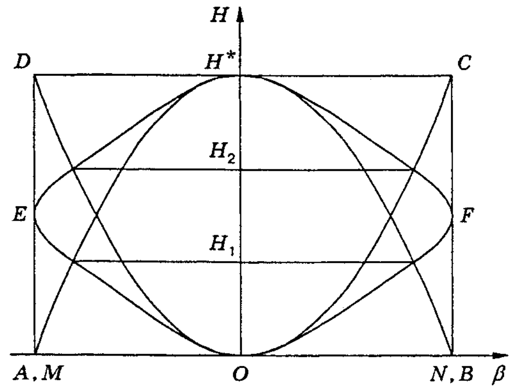

Fig. 3

System II.2 becomes well-posed for great depths of the lower layer satisfying the inequality $H_2 < H < H^*$, and, finally, system II.3 is well-posed for small depths of the upper layer satisfying the condition $0 < H < H_1$.

In conclusion, we determine how Fig. 1 transforms in the limits $\lambda \to 0$ and $\lambda \to 1$. As $\lambda \to 0$, the abscissas of the points A and B tend to $\mp\infty$, those of M and N tend to $\mp\sqrt{gH^*/3}$, and the abscissas of the points E, F, D, and C tend to zero. In this case,

$$\lim_{\lambda \to 0} H_2(\lambda) = H^*, \qquad \lim_{\lambda \to 0} H_1(\lambda) = \frac{4\sqrt{29} - 17}{35} H^* \simeq 0.13 H^*$$

(as an example, the case of $\lambda = 0, 1$ is shown in Fig. 2). As $\lambda \to 1$, the abscissas of the points A, M, E, and D tend to $-\sqrt{gH^*}$, those of the points B, N, F, and C tend to $\sqrt{gH^*}$, while the ordinates of the points E and F tend to $H^*/2$ and $\lim_{\lambda \to 1} H_1(\lambda) = H^*/3$ and $\lim_{\lambda \to 1} H_2(\lambda) = 2H^*/3$ (this limiting case is shown in Fig. 3).

The natural (at first glance) assumption that the limiting case shown in Fig. 3 must correspond to the model III of [1] in the limit

$$\mu \to 0, \qquad H \to H^* H, \qquad \beta \to \sqrt{\mu H^*}\beta, \qquad t \to t/\sqrt{\mu H^*} \tag{3.15}$$

turns out to be wrong. An explanation of this is that for model III, the equation of total energy (3.5) takes the form

$$e_t + [Hh(h - H)\beta^3]_x = 0,$$

where $e = Hh\beta^2$ and $h = 1 - H$, and, hence, the total energy e in terms of the basis variables $\beta = V - v$ and H (system II.1), $\bar{\beta} = V$ and H (system II.2), and $\tilde{\beta} = v$ and H (system II.3) is written as

$$e(\beta, H) = H(1 - H)\beta^2, \qquad e(\bar{\beta}, H) = \frac{H\bar{\beta}^2}{1 - H}, \qquad e(\tilde{\beta}, H) = \frac{1 - H}{H}\tilde{\beta}^2. \tag{3.16}$$

It is easy to see that all three functions (3.16) are nonconvex in the entire domain of hyperbolicity $\{0 < H < 1, |\beta| < g\}$ for model III.

Model I. For well-posed formulation of model I in the form of a complete system of conservation laws, the two laws of conservation of mass in the layers (3.1) and the law of conservation of total momentum conservation (3.4) must be used as the basis laws, and the total energy conservation law (3.5) must be employed as the closing conservation law. We first consider the system of basis conservation laws for which total momentum is not conserved.

System I.0. For system I.0, we choose the law of conservation of mass (3.1) and the laws of conservation of local momenta in the layers (3.2) as the basis conservation laws. Expressing the total energy (3.6) in terms of the basis variables V, v, H, and h of system I.0, we obtain the function

$$e(V,v,H,h) = HV^2 + \lambda hv^2 + g(H^2 + \lambda h^2 + 2\lambda hH),$$

which [similarly to its analog (2.6) for a single–layer fluid] is convex provided the flows in both layers are subcritical

$$|V| < \sqrt{gH}, \qquad |v| < \sqrt{gh} \tag{3.17}$$

and if the following condition [which strengthens (3.17)] is satisfied:

$$(Hg - V^2)(hg - v^2) > \lambda g^2 hH.$$

Obviously, these conditions are not compatible with the hyperbolicity condition obtained in [1] for model I, which imposes a restriction only on the difference of the velocities in the layers, and, therefore, system I.0 cannot be considered well-posed.

System I.1, I.2, and I.3. For systems I.1, I.2, and I.3, we use the two laws of conservation of mass in layers (3.1) and the law of conservation of total momentum (3.4) as the basis conservation laws. As the fourth basis conservation law, for system I.1 we take (by analogy with system II.2) the law of conservation of the jump of local momentum at the interface between the layers (3.7), for system I.2 we choose (by analogy with system II.1) the law of conservation of local momentum in the lower layer [the second of Eqs. (3.2)], and for system I.3 we take (by analogy with system II.3) the law of conservation of local momentum conservation in the upper layer [the first of Eqs. (3.2)].

Expressing the total energy (3.6) in terms of the basis variables α, β, H, and h of system I.1, we obtain

$$e(\alpha,\beta,H,h) = (b\alpha^2 + 2\mu Hh\alpha\beta + Ha\beta^2)/(H^*)^2 + g\varphi,$$

where $b = H + \lambda h$, $a = \lambda H + h$, $H^* = H + h$, and $\varphi = H^2 + \lambda h^2 + 2\lambda hH$. The necessary condition for convexity of this function $e_{HH} > 0$, written in the hyperplane $H^* = \text{const}$ for $\alpha = 0$ and $\lambda > 0.5$, is

$$\beta^2 = (V - \lambda v)^2 < \mu g(H^*)^2/[(2\lambda - 1)H + (2 - \lambda)h]. \tag{3.18}$$

Writing the total energy (3.6) in terms of the basis variables α, V, H, and h of system I.2 we obtain the function

$$e(\alpha,V,H,h) = (\alpha^2 - 2H\alpha V + HbV^2)/(\lambda h) + g\varphi,$$

for which the necessary convexity condition $e_{VV}e_{HH} - e_{VH}^2 > 0$ for $\alpha = 0$ is given by

$$V^2 < \lambda gHhb/(H^2 + Hb + b^2). \tag{3.19}$$

Writing the total energy (3.6) in terms of the basis variables α, v, H, and h of system I.3, we obtain the function

$$e(\alpha,v,H,h) = (\alpha^2 - 2\lambda h\alpha v + \lambda hbv^2)/H + g\varphi,$$

for which the necessary convexity condition $e_{vv}e_{hh} - e_{vh}^2 > 0$ for $\alpha = 0$ is given by

$$v^2 < gHhb/(h^2 + hb + b^2). \tag{3.20}$$

Since inequalities (3.18)–(3.20) are not compatible with the hyperbolicity condition for system I (which imposes a restriction only on the difference of the velocities in the layers), all three systems I.1, I.2, and I.3 (as well as system I.0) are not well-posed.

System I.4. An analysis of systems I.1, I.2, and I.3 shows that the necessary condition for convexity of the corresponding function leads to a restriction on the fourth, additional basis function. Since this function does not coincide with the velocity difference in the layers (which enters in the hyperbolicity condition for system I), the convexity conditions obtained turn out to be incompatible with the hyperbolicity condition. This leads to the assumption that to obtain a well-posed complete system of conservation laws for model I,

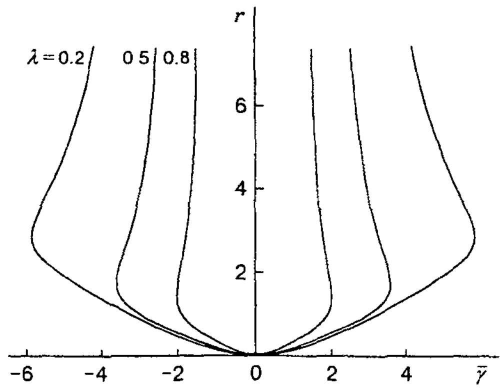

Fig. 4

it is necessary to supplement the laws of conservation of mass in the layers (3.1) and total momentum (3.4)] by the basis law of conservation of velocity jump at the interface between the layers:

$$\gamma_t + [(V^2 - v^2)/2 - \mu g h]_x = 0,$$

where $\gamma = V - v$ (this system is denoted by I.4). We note that the same additional basis conservation law appears in the limiting case for the models proposed in [14, 15].

Writing the total energy (3.6) in terms of the basis variables α, γ, H, and h of system I.4, we obtain the function

$$e(\alpha, \gamma, H, h) = (\alpha^2 + \lambda h H \gamma^2)/b + g\varphi,$$

which is convex provided that

$$|\bar{\gamma}| < f(\lambda, r) = \frac{\bar{b}}{\lambda r}\sqrt{\frac{g}{2}\left(c - \sqrt{c^2 - 4\lambda^2 \mu r^3}\right)}, \tag{3.21}$$

where $\bar{b} = 1 + \lambda r$, $c = 1 + \lambda^2 r^3$ and $\bar{\gamma} = \gamma/\sqrt{H} = (V - v)/\sqrt{H}$. Here and $r = h/H$ are the same variables as the ones used in [1] to formulate the hyperbolicity condition for model I. In this case, the function $f(\lambda, r)$ has the limits

$$\lim_{r \to 0} f(\lambda, r) = 0, \qquad \lim_{r \to \infty} f(\lambda, r) = \sqrt{\mu g}/\lambda.$$

In Fig. 4, the convexity domain (3.21) is shown for $\lambda = 0.2$, 0.5, and 0.8 and $g = 10$. This domain is entirely contained in one of the two unconnected subdomains of the hyperbolicity domain for model I, i.e., the subdomain that corresponds to small values of the velocity jump γ, for which the flow is more stable from a physical viewpoint.

Thus, the convexity condition (3.21) is in good agreement with the hyperbolicity condition for model I, and, hence, the complete system of conservation laws is well-posed.

We note in conclusion that the complete system for model III, for which unique solvability of the problem of decay of an arbitrary discontinuity in the hyperbolicity domain was proved in [16] using the characteristic stability condition of [9], is the general limiting case (3.15) of the well-posed complete systems I.4 and II.1. We also note that the complete systems I.4 and II.1 can be extended to the cases of plane potential flows and multilayer shallow-water flows.

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